Use the explicit method

$$u_m^{n+1} = 2(1-p^2)u_m^n + p^2(u_{m-1}^n + u_{m+1}^n) - u_m^{n-1}$$

and central difference approximation for the derivative conditions, to calculate a solution for $0 \leqslant x \leqslant 1$ and $0 \leqslant t \leqslant 0.5$ with $h=k=.1$.

8. The first and second *Lees* ADI methods for solving the equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

can be written as

(i) $u^{*n+1}_{l,m} = 2u^n_{l,m} - u^{n-1}_{l,m} + p^2\delta_x^2[\eta u^{*n+1}_{l,m} + (1-2\eta)u^n_{l,m} + \eta u^{n-1}_{l,m}]$
$\qquad + p^2\delta_y^2[(1-2\eta)u^n_{l,m} + 2\eta u^{n-1}_{l,m}]$
$u^{n+1}_{l,m} = u^{*n+1}_{l,m} + p^2\eta\delta_y^2(u^{n+1}_{l,m} - u^{n-1}_{l,m})$

and

(ii) $u^{*n+1}_{l,m} = 2u^n_{l,m} - u^{n-1}_{l,m} + p^2\delta_x^2[\eta u^{*n+1}_{l,m} + (1-2\eta)u^n_{l,m} + \eta u^{n-1}_{l,m}] + p^2\delta_y^2 u^n_{l,m}$
$u^{n+1}_{l,m} = u^{*n+1}_{l,m} + \eta p^2\delta_y^2(u^{n+1}_{l,m} - 2u^n_{l,m} + u^{n-1}_{l,m})$

where η is arbitrary.

Determine the uniform difference schemes in (i) and (ii). Show that the principal parts of the truncation error and the stability criteria are the same for both methods.

9. Write the first and second one parameter *Lees* ADI methods for the solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + cu$$

Determine the order of accuracy and the stability criterion for both methods.

10. The first and second *Lees* ADI methods for the equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

are of the form

(i) $u^{*n+1} = 2u^n - u^{n-1} + p^2\delta_x^2[\eta u^{*n+1} + (1-2\eta)u^n + \eta u^{n-1}]$
$\qquad + p^2(\delta_y^2 + \delta_z^2)[(1-2\eta)u^n + 2\eta u^{n-1}]$
$u^{**n+1} = u^{*n+1} + p^2\eta\delta_y^2(u^{**n+1} - u^{n-1})$
$u^{n+1} = u^{**n+1} + p^2\eta\delta_z^2(u^{n+1} - u^{n-1})$

and

(ii) $u^{*n+1} = 2u^n - u^{n-1} + p^2\delta_x^2[\eta u^{*n+1} + (1-2\eta)u^n + \eta u^{n-1}] - p^2(\delta_y^2 + \delta_z^2)u^n$
$u^{**n+1} = u^{*n+1} + \eta p^2\delta_y^2(u^{**n+1} - 2u^n + u^{n-1})$
$u^{n+1} = u^{**n+1} + \eta p^2\delta_z^2(u^{n+1} - 2u^n + u^{n-1})$

central grid point. Accurate results are obtained for $p=1$, 4 and 8 when β/α lies in the range $6.0 \leqslant \beta/\alpha \leqslant 8.0$, $1.3 \leqslant \beta/\alpha \leqslant 1.6$ and $1.1 \leqslant \beta/\alpha \leqslant 1.25$ respectively. It is seen that for fixed p, a value of β/α in the given range can be found which has an accuracy better than the results given in Table 6.8. The errors in the solution using the Beam-Warming method ($\alpha=\beta$), are higher than the results obtained here.

Bibliographical Note

The excellent texts dealing with the numerical solutions of the hyperbolic equations are 9, 96, 184 and 203. The stability of the linear finite difference equations is discussed in 168. The high order difference schemes are given in 80, 126 and 129. The difference schemes for the second order hyperbolic differential equations with variable coefficients and with or without mixed derivatives are studied in 44, 169, 180 and 181. The solution of one dimensional wave equation under derivative boundary conditions has been examined in 150.

The LOD method for obtaining the numerical solution of the hyperbolic equations in two and three space dimensions is given in 98, 130 and 215.

The explicit and implicit difference schemes for the system of hyperbolic equations are discussed in 2, 96, 99, 116, 159, 167, 176, 178, 179, 202, 204, 210, 225, 232 and 239. The *Kreiss* stability analysis of the difference schemes is given in 1, 16, 90, 95, 105 and 158.

I roblems

1. The function $u(x, t)$ satisfies the differential equation

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}+cu$$

with boundary conditions

$$u=0 \text{ for } x=0 \text{ and } x=1,\ t \geqslant 0$$

Let u and $\partial u/\partial t$ be prescribed for $t=0$, $0 \leqslant x \leqslant 1$.

(i) Derive the difference scheme by replacing the derivatives by central differences.

(ii) Obtain the principal part of the truncation error.

(iii) Determine the stability criterion of the difference scheme.

2. The differential equation

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}+cu$$

is approximated by the difference scheme

$$(1+\tau\delta_t^2)^{-1}\delta_t^2 u_m^n=p^2\delta_x^2 u_m^n+cp^2h^2u_m^n$$

where τ is arbitrary, $p=k/h$ and c is a constant.

where u^{*n+1} and u^{**n+1} are intermediate values and η is arbitrary. Obtain the principal part of the truncation errors and the conditions for unconditional stability for both methods.

11. The second order hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2}=a\frac{\partial^2 u}{\partial x^2}+b\frac{\partial u}{\partial x}+cu,\ 0\leqslant x\leqslant 1$$

with the initial and boundary conditions

$$u(x,0)=g_1(x),\ \frac{\partial u(x,0)}{\partial t}=g_2(x)$$

$$u(0,t)=f_1(t),\qquad u(1,t)=f_2(t)$$

where a, b, c are constants, $a>0$, and $f_1(t)$, $f_2(t)$, $g_1(x)$ and $g_2(x)$ are known functions, is solved using the difference scheme

$$\begin{aligned}&(1-\beta_1\theta)u_{m-1}^{n+1}+4(1+\beta_2\theta)u_m^{n+1}+(1-\beta_3\theta)u_{m+1}^{n+1}\\&=[2+\beta_1(1-2\theta)]u_{m-1}^n+4[2-\beta_2(1-2\theta)]u_m^n+[2+\beta_3(1-2\theta)]u_{m+1}^n\\&\quad-(1-\beta_1\theta)u_{m-1}^{n-1}-4(1+\beta_2\theta)u_{m-1}^{n-1}-(1-\beta_3\theta)u_{m+1}^{n-1}\end{aligned}$$

where

$$\beta_1=6ap^2-3bpk+ck^2,\ \beta_2=3ap^2-ck^2$$
$$\beta_3=6ap^2+3bpk+ck^2,$$
$$p=k/h,\ 0\leqslant\theta\leqslant 1$$

Obtain:

(i) the local truncation error;
(ii) the stability criterion.

12. Let $u_m^{n+1}=u_m^n+\frac{1}{2}p(4u_{m+1}^n-u_{m+2}^n-3u_m^n)+\frac{1}{2}p^2(u_{m+2}^n-2u_{m+1}^n+u_m^n)$

where $p=k/h$, be a difference approximation to the differential equation

$$\frac{\partial u}{\partial t}=\frac{\partial u}{\partial x}$$

(i) What does the schematic form look like?
(ii) How big is the local truncation error?
(iii) State some values of p for which the difference approximation is stable. (BIT 9 (1969), 400)

13. The differential equation

$$\frac{\partial u}{\partial t}=b\frac{\partial u}{\partial x}-cu,\ c>0$$

is approximated by the difference equation

$$u_m^{n+1}=u_m^{n-1}+2pb\mu_x\delta_x u_m^n-2kcu_m^n$$

where $p=k/h$.

20. Consider the following three-step method

$$\mathbf{u}^{n+1/2}_{m+1/2} = \frac{1}{2}(\mathbf{u}^n_{m+1} + \mathbf{u}^n_{m-1/2}) + \frac{p}{2}(\mathbf{f}^n_{m+1} - \mathbf{f}^n_m)$$

$$\mathbf{u}^{n+1}_m = \mathbf{u}^n_m + p(\mathbf{f}^{n+1/2}_{m+1/2} - \mathbf{f}^{n+1/2}_{m-1/2})$$

$$\mathbf{u}^{n+2}_m = \mathbf{u}^n_m + p(\mathbf{f}^{n+1}_{m+1} - \mathbf{f}^{n+1}_{m-1})$$

for the solution of the differential equation

$$\frac{\partial \mathbf{u}}{\partial t} = \frac{\partial \mathbf{f}}{\partial x} = \mathbf{A}\frac{\partial \mathbf{u}}{\partial x}$$

where $\mathbf{A} = \dfrac{\partial \mathbf{f}}{\partial \mathbf{u}}$.

(i) Determine the amplification matrix **G** associated with this method for **A** constant matrix.
(ii) If **A** is diagonalizable then find the stability condition.
(iii) Is the three-step method dissipative in the sense of *Kreiss*?
(iv) Determine the relative phase error.
(v) Consider a four-step-method by adding another leapfrog step to the three-step method

$$\mathbf{u}^{n+3}_m = \mathbf{u}^{n+1}_m + p(\mathbf{f}^{n+2}_{m+1} - \mathbf{f}^{n+2}_{m-1}).$$

Show that the fourth-step method is dissipative of order four.

21. Consider the leapfrog scheme of the form

$$\mathbf{u}^{n+1}_m = \mathbf{u}^{n-1}_m + p(\mathbf{f}^n_{m+1} - \mathbf{f}^n_{m-1}) - \frac{\sigma}{16}(\mathbf{u}^{n-1}_{m-2} - 4\mathbf{u}^{n-1}_{m-1} + 6\mathbf{u}^{n-1}_m - 4\mathbf{u}^{n-1}_{m+1} + \mathbf{u}^{n-1}_{m+2})$$

where σ is an arbitrary parameter, for the solution of the equation

$$\frac{\partial \mathbf{u}}{\partial t} = \frac{\partial \mathbf{f}}{\partial x}$$

Determine the stability criterion when $\mathbf{f} = \mathbf{A}\mathbf{u}$ where **A** is a constant matrix.

22. We assume the difference scheme

$$\mathbf{u}^{*n+1}_m = \frac{1}{2}(\mathbf{u}^n_{m+1} + \mathbf{u}^n_{m-1}) - ap(\mathbf{f}^n_{m+1} - \bar{\mathbf{f}}^n_{m-1} - 2h\mathbf{z}^n_m)$$

$$\mathbf{u}^{n+1}_m = \mathbf{u}^n_m - \frac{p}{4}\Big[\Big(1 - \frac{1}{2a}\Big)\mathbf{f}^n_{m+1} - \mathbf{f}^n_{m-1} - 2h\mathbf{z}^n_m) + \frac{1}{2a}(\mathbf{f}^{*n+1}_{m+1} - \mathbf{f}^{*n+1}_{m-1} - 2h\mathbf{z}^{*n+1}_m) + (\bar{\mathbf{f}}^{n+1}_{m+1} - \bar{\mathbf{f}}^{n+1}_{m-1} - 2h\bar{\mathbf{z}}^{n+1}_m)\Big]$$

where a is an arbtirary constant to be prescribed and

$$\mathbf{f}_m^{*n+1} = \mathbf{f}(\mathbf{u}_m^{*n+1}, x_m, t_n)$$
$$\mathbf{z}_m^{*n+1} = \mathbf{z}(\mathbf{u}_m^{*n+1}, x_m, t_n)$$
$$\overline{\mathbf{f}}_m^{n+1} = \mathbf{f}(\mathbf{u}_m^n, x_m, t_{n+1})$$
$$\overline{\mathbf{z}}^{n+1} = \mathbf{z}(\mathbf{u}_m^n, x_m, t_{n+1})$$

for the solution of the nonlinear system

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{u}, x, t)}{\partial x} = \mathbf{z}(\mathbf{u}, x, t)$$

with the initial and boundary conditions

$$\mathbf{u}(x, 0) = \mathbf{u}_0(x)$$
$$\mathbf{u}(0, t) = \mathbf{u}_1(x) \quad , \quad t > 0$$

(i) How big is the local truncation error for arbitrary a?

(ii) State some values of a for which the difference scheme is stable.

23. Solve the following initial and boundary value problems;

(i) $\dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} = 0, \qquad 0 < x < \infty$

$$u(x, 0) = f(x), \quad 0 < x \leqslant 10$$

where

$$f(x) = \begin{cases} 0 & , \quad 0 \leqslant x \leqslant 1 \\ \sin 8\pi(x-1) & , \quad 1 \leqslant x \leqslant 2 \\ 0 & , \quad 2 \leqslant x < \infty \end{cases}$$

with the exact solution, $u(x, t) = f(x-t)$

(ii) $\dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}\left(\dfrac{1}{2} u^2\right) = 0$

$$u(x, 0) = 1 + x$$
$$u(0, t) = 1/(1+t)$$

with the exact solution

$$u(x, t) = (1+x)/(1+t)$$

(iii) $\dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}\left(\dfrac{1}{2} u^2\right) = 0$

$$u(x, 0) = \begin{cases} 1 & , \quad 0 \leqslant x \leqslant .1 \\ 0 & , \quad x > .1 \end{cases}$$
$$u(0, t) = 1 \quad , \quad t > 0$$

using the methods given in the text. Obtain the numerical results from the computation.

then show that the amplification matrix **G** is given by

$$\mathbf{G} = \mathbf{I} + 4i\mathbf{N} - 8\mathbf{N}^2 - 8i\mathbf{N}\left[\mathbf{A}^2 \sin^2\frac{\beta h}{2} + (\mathbf{AB} + \mathbf{BA}) \sin\frac{\beta h}{2}\cos\frac{\beta h}{2}\right.$$

$$\left.\sin\frac{\gamma h}{2}\cos\frac{\gamma h}{2} + \mathbf{B}^2 \sin^2\frac{\gamma h}{2}\right]$$

When **A** and **B** commute prove that the eigenvalues of **G** satisfy

$$|g|^2 = 1 - 64n^2\left[\lambda_A^2 \sin^4\frac{\beta h}{2} + \lambda_B^2 \sin^4\frac{\gamma h}{2}\right.$$

$$\left. - \left(\lambda_N^2 + \lambda_A^2 \sin^4\frac{\beta h}{2} + \lambda_B^2 \sin^2\frac{\gamma h}{2}\right)^2\right]$$

where λ_N, λ_A, λ_B are corresponding eigenvalues of **N**, **A**, **B** respectively. The stability criterion is satisfied if

$$[\rho\rho(\mathbf{A})]^{2/3} + [\rho\rho(\mathbf{B})]^{2/3} \leqslant 1$$

28. Consider the difference scheme

$$\mathbf{u}_{l,m}^{n+1} = \left[\mathbf{I} - \mathbf{A}^2 - \mathbf{B}^2 + \frac{1}{2}(\mathbf{AB} + \mathbf{BA})\right]\mathbf{u}_{l,m}^n$$

$$+ \left[-\frac{1}{2}\mathbf{A} + \frac{1}{2}\mathbf{A}^2 - \frac{1}{4}(\mathbf{AB} + \mathbf{BA}) - \mathbf{\Lambda}\right]\mathbf{u}_{l-1,m}^n$$

$$+ \left[\frac{1}{4}(\mathbf{AB} + \mathbf{BA}) + \mathbf{\Lambda}\right]\mathbf{u}_{l-l,m-1}^n$$

$$+ \left[\frac{1}{2}\mathbf{B} + \frac{1}{2}\mathbf{B}^2 - \frac{1}{4}(\mathbf{BA} + \mathbf{BA}) + \mathbf{\Lambda}\right]\mathbf{u}_{l,m+1}^n$$

$$+ \left[-\frac{1}{2}\mathbf{B} + \frac{1}{2}\mathbf{B}^2 - \frac{1}{4}(\mathbf{AB} + \mathbf{BA}) - \mathbf{\Lambda}\right]\mathbf{u}_{l,m-1}^n$$

$$+ \left[\frac{1}{4}(\mathbf{AB} + \mathbf{BA}) - \mathbf{\Lambda}\right]\mathbf{u}_{l+1,m+1}^n$$

$$+ \left[\frac{1}{2}\mathbf{A} + \frac{1}{2}\mathbf{A}^2 - \frac{1}{4}(\mathbf{AB} + \mathbf{BA}) + \mathbf{\Lambda}\right]\mathbf{u}_{l+1,m}^n$$

where $\mathbf{\Lambda}$ is an arbitrary $N \times N$ matrix, for the solution of the differential equation

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{A}\frac{\partial \mathbf{u}}{\partial x} + \mathbf{B}\frac{\partial \mathbf{u}}{\partial y}$$

Show that the amplification matrix, for $\mathbf{\Lambda} = -\frac{1}{4}(\mathbf{A} + \mathbf{B})$, is given by

$$\mathbf{G} = \mathbf{I} - \mathbf{A}^2(1 - \cos\beta h) - \mathbf{B}^2(1 - \cos\gamma h)$$

$$+ \frac{1}{2}(\mathbf{AB} + \mathbf{BA})[(1 - \cos\gamma h)(1 - \cos\beta h) - \sin\beta h \sin\gamma h]$$

$$+\frac{i}{2}[\mathbf{A}(\sin(\beta h+\gamma h)+\sin\beta h-\sin\gamma h)+\mathbf{B}(\sin(\beta h+\gamma h)$$
$$-\sin\beta h+\sin\gamma h)]$$

29. Solve the initial and boundary value problems:

(i) $\dfrac{\partial u}{\partial t}=\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}$

$$u(x, y, 0)=\sin 2\pi x \sin 2\pi y$$
$$u(1, y, t)=\sin 2\pi t \sin 2\pi(y+t)$$
$$u(x, 1, t)=\sin 2\pi t \sin 2\pi(x+t)$$

with the exact solution

$$u(x, y, t)=\sin 2\pi(x+t)\sin 2\pi(y+t)$$

on $\bar{G}=\{[0\leqslant x\leqslant 1]\times[0\leqslant y\leqslant 1]\}[t>0]$;

(ii) $\dfrac{\partial u}{\partial t}+\dfrac{\partial}{\partial x}\left(\dfrac{1}{4}u^2\right)+\dfrac{\partial}{\partial y}\left(\dfrac{1}{4}u^2\right)=0$

$$u(x, y, 0)=\frac{1}{4}(x+y)^2$$
$$u(0, y, t)=\left\{\frac{1-(1+yt)^{1/2}}{t}\right\}^2$$
$$u(x, 0, t)=\left\{\frac{1-(1+xt)^{1/2}}{t}\right\}^2$$

with the exact solution

$$u(x, y, t)=\left[\frac{1-(1+(x+y)t)^{1/2}}{t}\right]^2$$

on $\bar{G}=\{[0\leqslant x\leqslant 1]\times[0\leqslant y\leqslant 1]\}\times[t>0]$;

using the numerical methods given in the text.

30. Consider the first order hyperbolic system in three space dimensions

$$\frac{\partial \mathbf{u}}{\partial t}+\mathbf{A}\frac{\partial \mathbf{u}}{\partial x}+\mathbf{B}\frac{\partial \mathbf{u}}{\partial y}+\mathbf{C}\frac{\partial \mathbf{u}}{\partial z}=\mathbf{0}$$

with appropriate initial and boundary condition, where **A**, **B** and **C** are constant matrices and **u** is a vector function.
Write:

(i) the diffusing scheme;
(ii) the leapfrog scheme;
(iii) the two-step Lax-Wendroff scheme and simplify to get a composite scheme.

7

Difference Methods for Elliptic Partial Differential Equations

7.1 INTRODUCTION

The elliptic partial differential equations always occur purely as boundary value problems. Thus the considerations of Chapter 4 can be extended in a natural way to the boundary value problems in more than one variable. The applications of the difference methods to the elliptic differential equations often lead to a large system of algebraic equations and their solution is a major problem in itself. The iterative methods are generally used to solve the large system of equations.

Alternatively, if the solution of an elliptic boundary value problem is interpreted as the stationary solution of an appropriate initial boundary value problem, then such problems can be treated by the methods given in Chapters 5 and 6.

Here we shall discuss difference schemes to solve numerically linear elliptic boundary value problems.

7.2 DIFFERENCE SCHEMES

Consider the solution of the differential equation

$$Lu=a\frac{\partial^2 u}{\partial x^2}+c\frac{\partial^2 u}{\partial y^2}+d\frac{\partial u}{\partial x}+e\frac{\partial u}{\partial y}+fu=f^* \qquad (7.1)$$

in the region $\mathcal{R}$ subject to the Dirichlet boundary condition

$$u=g(x, y) \text{ on } \partial\mathcal{R} \qquad (7.2)$$

where a, c, d, e, f and f^* are functions of x and y. We assume that these functions are continuous in $\mathcal{R}+\partial\mathcal{R}$ and furthermore $a > 0$, $c > 0$, $f \leqslant 0$.

Let us superimpose on $\mathcal{R}$ a rectangular network with mesh lengths h and k in the x and y-directions, respectively. The nodal points are given by

$$x_l=x_0+lh, \quad l=0, \pm 1, \pm 2, \cdots$$
$$y_m=y_0+mk, \quad m=0, \pm 1, \pm 2, \cdots$$

Two nodal points are called *neighbouring* points if they are one mesh length apart along the x or y axis. A nodal point is called a pivot if it lies within or on $\partial\mathcal{R}$. A pivot is termed as an *internal* pivot if it has four neighbouring pivots. A *boundary* pivot is one for which at least one of its neighbouring nodal points is not a pivot. The region $\mathcal{R}$ and its boundary curve $\partial\mathcal{R}$ are shown in Figure 7.1. The internal pivots comprising $\mathcal{R}$ are denoted by shaded circles and the boundary pivots comprising $\partial\mathcal{R}$ by open circles. The pivot

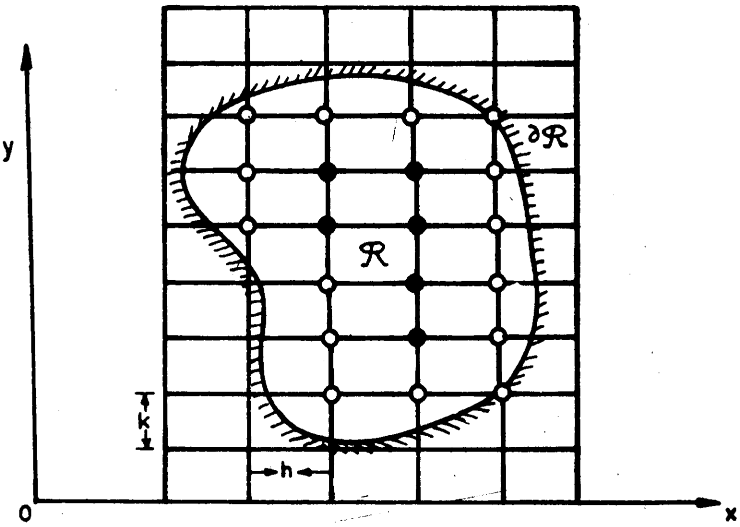

Fig. 7.1 The region $\mathcal{R}$ and its boundary curve $\partial\mathcal{R}$

(x_l, y_m) will be denoted by (l, m). We can now form the difference scheme by substituting difference expressions for the derivatives in (7.1). The difference scheme at every internal pivot (l, m) can be written as

$$L_h u_{l,m} = a_{l,m}\frac{\delta_x^2 u_{l,m}}{h^2} + c_{l,m}\frac{\delta_y^2 u_{l,m}}{k^2} + d_{l,m}\frac{\mu_x\delta_x u_{l,m}}{h} + e_{l,m}\frac{\mu_y\delta_y u_{l,m}}{k} + f_{l,m}u_{l,m} = f^*_{l,m} \tag{7.3}$$

where $u_{l,m}$ is the approximate value of $u(x_l,y_m)$. μ_x is the average operator in x direction and $a_{l,m}$, $c_{l,m}, \cdots$ are the values of the coefficients at (l, m).

Simplifying (7.3), we obtain the difference scheme

$$L_h u_{l,m} = \frac{1}{h^2}(A_{l,m}u_{l+1,m} + B_{l,m}u_{l-1,m} + C_{l,m}u_{l,m+1} + D_{l,m}u_{l,m-1} - E_{l,m}u_{l,m}) = f^*_{l,m} \tag{7.4}$$

where

$$A_{l,m} = a_{l,m} + \tfrac{1}{2}hd_{l,m}$$
$$B_{l,m} = a_{l,m} - \tfrac{1}{2}hd_{l,m}$$

$$C_{l,m} = \alpha c_{l,m} + \tfrac{1}{2}\alpha^{1/2} h e_{l,m}$$
$$D_{l,m} = \alpha c_{l,m} - \tfrac{1}{2}\alpha^{1/2} h e_{l,m}$$
$$E_{l,m} = 2a_{l,m} + 2\alpha c_{l,m} - h^2 f_{l,m}$$
$$\alpha = \frac{h^2}{k^2}$$

If (l, m) is a boundary pivot such that it lies on $\partial\mathcal{R}$, then we substitute $u_{l,m}$ by its value $g_{l,m}$. However, if the boundary of the regions $\mathcal{R}$ is not such that the network can be drawn to have the boundary coincide with boundary pivots, then we must proceed differently at boundary pivots near the boundary $\partial\mathcal{R}$.

Let us consider the general case of a group of five points whose spacing is nonuniform, $h\delta_1$ and $h\delta_3$ along x axis, $k\delta_2$ and $k\delta_4$ along y axis, arranged as in Figure 7.2. We represent

$$\begin{aligned} u_0 &= u(x_l, y_m) \quad , \quad u_1 = u(x_l + \delta_1 h, y_m) \\ u_2 &= u(x_l, y_m + \delta_2 k) \quad , \quad u_3 = u(x_l - \delta_3 h, y_m) \\ u_4 &= u(x_l, y_m - \delta_4 k) \end{aligned} \tag{7.5}$$

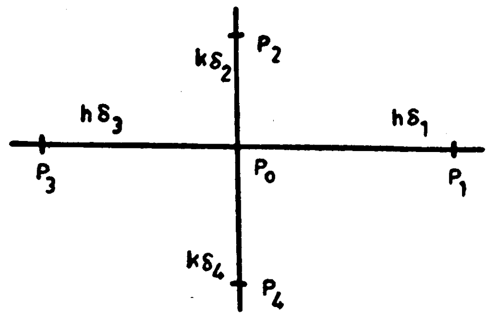

Fig. 7.2 The nodal points at unequal distances

The first and second derivatives can be written as

(i) $\dfrac{\partial u}{\partial x} = \dfrac{u_1 - u_0}{\delta_1 h} + 0(h)$

(ii) $\dfrac{\partial u}{\partial x} = \dfrac{u_0 - u_3}{\delta_3 h} + 0(h)$

(iii) $\dfrac{\partial u}{\partial y} = \dfrac{u_2 - u_0}{\delta_2 k} + 0(k)$

(iv) $\dfrac{\partial u}{\partial y} = \dfrac{u_0 - u_4}{\delta_4 k} + 0(k)$

(v) $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{2}{h^2}\left[\dfrac{1}{(\delta_1 + \delta_3)}\left(\dfrac{u_1}{\delta_1} + \dfrac{u_3}{\delta_3}\right) - \dfrac{u_0}{\delta_1 \delta_3}\right] + 0(h)$

(vi) $\dfrac{\partial^2 u}{\partial y^2} = \dfrac{2}{k^2}\left[\dfrac{1}{(\delta_2 + \delta_4)}\left(\dfrac{u_2}{\delta_2} + \dfrac{u_4}{\delta_4}\right) - \dfrac{u_0}{\delta_2 \delta_4}\right] + 0(k)$ (7.6)